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Bilinearization of multidimensional topological magnets

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Abstract. A classical magnetic model with compact $su(2)$ and non-compact $su(1,1)$ spin phase space admitting the Hirota bilinear form is presented in an arbitrary number of space dimensions. The essential point of the construction is the presence of a velocity field with non-trivial vorticity tensor related to a topological current density. The model modifies in particular a variety of familiar topological equations, such as the Heisenberg ferromagnets and the $(2+1)$-dimensional $O(3)$ σ-model. Using the bilinear representation, several special cases and exact solutions of physical interest (spin waves, domain walls and vortices) are considered.

1. Introduction

In field theories governed by integrable nonlinear differential equations, an important role is played by those kinds of solutions, such as for example, solitons and vortices, which find applications in different physical areas [1]. We recall, for instance, the recent discovery of localized solitons in $2+1$ dimensions [2] and the studies made to extend their search in higher dimensions [3]. One of the most interesting aspects of the dynamics of these solutions is that they can simulate inelastic scattering processes of quantum particles as creation and annihilation, fusion and fission, and interactions with virtual particles [2, 4]. Therefore, we notice that in connection with the complete integrability of a given nonlinear field model, the existence of a linear problem associated with the underlying equation of motion is crucial. This feature may help to build up the corresponding σ-model representation via the gauge equivalence theory [9]. On the other hand, special attention should be paid to those nonlinear σ-models, which are endowed with topological structures [6]. Concerning these models, many effects have been shown in the quantization of planar localized solutions. The latter have a fractional or even irrational spin and obey peculiar statistics, which are intermediate between the Bose–Einstein and Fermi–Dirac statistics [7]. These phenomena occur in the treatment of the quantum Hall effect and high temperature superconductivity [8]. Some attempts also exist to describe topological localized solutions in $3+1$-dimensional systems by introducing the Chern–Simons and Hopf invariants [9].

The above considerations suggest that any construction of physical nonlinear models, possibly with non-trivial topological structures, allowing analytical studies of the dynamics of solutions would be appreciated.

Following this line of research, here we present and investigate a multidimensional classical nonlinear magnet model with compact (sphere S^2) and non-compact (pseudosphere $S^{1,1}$) spin phase space, having a velocity field with a vorticity related to a topological charge density. Our model modifies in some interesting well known nonlinear field systems such as the Heisenberg ferromagnets, their generalization to anisotropic crystals with magnetic ordering depending on the spacial directions, the nonlinear σ-models, the Ishimori and the Ernst equation and many others [1, 10–13].

The main purpose of this work is to study the equations of the model by means of the Hirota bilinearization technique [14]. Adopting this procedure, we obtain exact solutions of domain wall type, spin waves and vortices. In section 2 we describe our model and discuss some reductions corresponding to particular space dimensions. In section 3 we write our model in a bilinearized form, while in section 4 we present examples of exact solutions. Finally, section 5 contains some concluding remarks.

2. The model

In order to formulate our model, let us consider a D-dimensional pseudo-Euclidean space $R^{p,q}$ $(D=p+q)$ with metric tensor $g_{\mu\nu}=\mathrm{diag}(+,\ldots,+,-,\ldots,-)$. The model describes the time evolution of the unit 'spin' vectors $\mathbf{S}=(S_1,S_2,S_3)$ according to the Landau–Lifshitz equation [15] in a local moving frame with velocity $v_u(x_\mu,t)$, where $S_j=S_j(x_\mu,t)$ $(x_\mu=(x_1,\ldots,x_D))$ and $S_3^2+\kappa^2(S_1^2+S_2^2)=1$. $\mathbf{S}$ belongs to the sphere S^2, or to the pseudosphere $S^{1,1}$, when $\kappa^2=1$ or $\kappa^2=-1$, respectively. The vorticity tensor of the velocity field v_μ is assumed to be related to the gradient of the spin field vector as follows

$$\mathbf{S}_t+v^\mu\partial_\mu\mathbf{S}=\mathbf{S}\times\partial^\mu\partial_\mu\mathbf{S} \tag{2.1a}$$

$$\partial_\mu v_\nu-\partial_\nu v_\mu=2\kappa^2\mathbf{S}\cdot(\partial_\mu\mathbf{S}\times\partial_\nu\mathbf{S}) \tag{2.1b}$$

where $\mu,\nu=1,2,\ldots,D$. Here $(\mathbf{a}\times\mathbf{b})_l=f_{lmn}a_m b_n$ is the exterior product associated with the structure constants f_{lmn} of the $su(2)$ and $su(1,1)$ algebras, $\partial_\nu=\partial/\partial x_\nu$ and $\partial^\mu=g^{\mu\nu}\partial_\nu$.

The interaction between the spin and the velocity fields is suggested by several examples concerning integrable models, like the Ishimori model [11], the Davey–Stewartson equation [16] and by the Papanicolau equation [13]. On the other hand, we recall that equation (2.1b) defines essentially the topological current pertinent to the $O(3)$ nonlinear σ-model [6], which implies a conserved topological charge for $D=2$ and 3.

In the context of the nonlinear σ-model, equation (2.1b) is invariant under a local gauge transformation $v'_\mu=v_\mu+\partial_\mu\alpha$ (being $\alpha=\alpha(x_\mu,t)$, any differentiable function). Thus one can interpret the velocity field v_μ as a gauge potential, leading to a new topological conserved quantity, the Hopf invariant, which describes the homotopy classes $\pi_3(S^2)=Z$ [17]. So one is able to handle the spin and the statistics of the topological configurations (skyrmions) of the model [6], However, since (2.1a) includes explicitly the field v_μ, our model is not invariant under the above-mentioned gauge transformation.

Finally, we notice that (2.1b) is inspired by the Mermin–Ho relation [18], which occurs in the theory of quantized vortices in the superfluid ^{3}He.

In order to demonstate the great versatility of the model proposed here, we shall discuss some special cases, which can be obtained from (2.1) and from their different versions derived in section 3. In brief, we shall limit ourselves to a few examples in $D>1$ dimensions. A systematic treatment of all the cases will be presented elsewhere.

2.1. *Models in $D=2$ dimensions*

In the static limit for $v^\mu=0$, equation (2.1*a*) provide the (integrable) $su(2)$ and $su(1,1)$ nonlinear σ-models in two-dimensional Minkovskian or Euclidean space, respectively, according to the metric tensor $g^{\mu\nu}$ [20]. Conversely, the case $v_\mu=$ constant was considered by Papanicolau [13], in connection with the study of the static configurations of planar ferromagnets. In [13] the Lax pair of the system and a duality transformation were found.

Let us suppose that the two-dimensional velocity field is represented in terms of a real scalar field ϕ, namely

$$v_x=\partial_y\phi \qquad v_y=\alpha^2\partial_x\phi \qquad (\alpha^2=\pm 1) \tag{2.2}$$

with metric tensor $g^{\mu\nu}=\mathrm{diag}(1,\alpha^2)$. Then system (2.1) reduces to the Ishimori model [11]. This model, both in the compact and the non-compact version [21], is important because it is the first example of a nonlinear spin field model on the plane, allowing a Lax pair formulation. It admits exact solutions, which are classified by an integer topological charge (localized solitons [22], vortex-like [11], closed string-like, and doubly periodic solutions [21]). Furthermore, one can show that, in general, the Ishimori model is gauge equivalent to the complex Davey–Stewartson equation [19]. However, for $\alpha^2=-1$ the Ishimori model is gauge equivalent to a reduced case of the so-called *DS*-II equation [9].

2.2. *Models in $D=3$ dimensions*

In the static limit, for $v_\mu=0$ equation (2.1*a*) provide the $su(2)$ and $su(1,1)$ three-dimensional nonlinear σ-model. Integrable reductions for this model are well known [1, 20, 23].

Let us assume that the spin field takes an axially symmetric configuration (that is $\mathbf{S}=\mathbf{S}(\rho,z)$, where $\rho=(x^2+y^2)^{1/2}$). Then in the $su(2)$ case we obtain the Heisenberg model [10]; on the other hand, in the $su(1,1)$ case we have the Ernst equation for axially symmetric gravitational fields [12]. This equation admits a Lax pair and soliton solutions [24].

Finally, we observe that when restricting ourselves to the field configurations satisfying the constraint

$$\mathbf{S}\times\partial^\mu\partial_\mu\mathbf{S}=0 \tag{2.3}$$

the equation (2.1*a*) is reduced to the Euler equation for the vorticity field $\boldsymbol{S}=\nabla\times\boldsymbol{v}$ [25], which can be put into a Hamiltonian form [26]. On the other hand, equation (2.3) can be regarded as the stationary Landau–Lifshitz equation for isotropic ferromagnets or, from another point of view, as the nonlinear σ-model. These equations have solutions of the vortex type at least in two dimensions (instantons). Then, if we assume that such static solutions depend parametrically on time according to the Euler equation, we obtain a vortex hydrodynamics for a perfect fluid, eventually incompressible if the supplementary condition $\partial^\mu v_\mu=0$ is satisfied. Actually, the total magnetization $\mathbf{M}=\int\mathbf{S}\,\mathrm{d}^Dx$ for (2.1) is conserved only if the last constraint on the velocity field is verified.

3. Bilinearization of the model

For our purposes, it is convenient to have (2.1) in a different form. We perform the stereographic projection of the spin vector $\mathbf{S}$ on the complex plane ζ by means of the relationships

$$S_+ = S_1 + iS_2 = \frac{2\zeta}{1+\kappa^2|\zeta|^2} \qquad S_3 = \frac{1-\kappa^2|\zeta|^2}{1+\kappa^2|\zeta|^2}. \tag{3.1}$$

So the equations of motion (2.1) take the form

$$i(\partial_t\zeta + v^\mu\partial_\mu\zeta) + \partial^2\zeta - 2\kappa^2\frac{\partial^\mu\zeta\partial_\mu\zeta}{1+\kappa^2|\zeta|^2}\bar{\zeta} = 0 \tag{3.2a}$$

$$\partial_\mu v_\nu - \partial_\nu v_\mu = -4i\kappa^2\frac{\partial_\mu\bar{\zeta}\partial_\nu\zeta - \partial_\nu\bar{\zeta}\partial_\mu\zeta}{(1+\kappa^2|\zeta|^2)^2} \tag{3.2b}$$

where the overline denotes the complex conjugate.

Now in order to derive the Hirota represenation of the model, we introduce two complex functions f and g such that

$$\zeta = g/f. \tag{3.3}$$

Then the 'projective' representation of the spin components (3.1) is given by

$$S_+ = \frac{2\bar{f}g}{\bar{f}f+\kappa^2\bar{g}g}, \qquad S_2 = \frac{\bar{f}f-\kappa^2\bar{g}g}{\bar{f}f+\kappa^2\bar{g}g}. \tag{3.4}$$

By substituting (3.3) into (3.2a), after some algebra we obtain a complicated equation, which can be put in the bilinearized form

$$\begin{aligned}(|f|^2-\kappa^2|g|^2)&\left\{(iD_t - D^2) + \sum_\mu\left[iv^\mu + 2\frac{D^\mu(\bar{f}\circ f+\kappa^2\bar{g}\circ g)}{|f|^2+\kappa^2|g|^2}\right]D_\mu\right\}(\bar{f}\circ g)\\ &-g\bar{f}\left\{(iD_t - D^2) + \sum_\mu\left[iv^\mu + 2\frac{D^\mu(\bar{f}\circ f+\kappa^2\bar{g}\circ g)}{|f|^2+\kappa^2|g|^2}\right]D_\mu\right\}(\bar{f}\circ f - \kappa^2\bar{g}\circ g) = 0\end{aligned} \tag{3.5}$$

where D_μ and D_t are the Hirota operators, defined by

$$\begin{aligned}&D_\mu^j D_t^k a(x_1, \ldots, x_D, t)\circ B(x_1, \ldots, x_D, t)\\ &\quad = (\partial_\mu - \partial_{\mu'})^j(\partial_t - \partial_{t'})^k a(x_1, \ldots, x_D, t)b(x_1', \ldots x_D, t')_{x=x', t=t'}\end{aligned} \tag{3.6}$$

$D^\mu = g^{\mu\nu}D_\nu$ and $D^2 = D^\mu D_\mu$.

Equation (3.5) is satisfied by putting

$$(iD_t - D^2)(\bar{f}\circ g) = 0 \tag{3.7a}$$

$$(iD_t - D^2)(\bar{f}\circ f - \kappa^2\bar{g}\circ g) = 0 \tag{3.7b}$$

while the velocity field is

$$v_\mu = 2i\frac{D_\mu(\bar{f}\circ f+\kappa^2\bar{g}\circ g)}{|f|^2+\kappa^2|g|^2}. \tag{3.8}$$

Since the expression (3.8) for the velocity field v_μ fulfils (3.2*b*), a solution (ζ, v_μ) of system (3.2) is obtained by solving (3.7) for f and g and using the definitions (3.3) and (3.8).

4. Special solutions

4.1. Special solutions in D = 2 dimensions

First we show some interesting particular solutions of the model in the case $D=2$. Now the bilinear system (3.7) to (3.8) takes the form

$$(\mathrm{i}D_t - D_x^2 - \alpha^2 D_y^2)(\bar{f}\circ g) = 0 \tag{4.1a}$$

$$(\mathrm{i}D_t - D_x^2 - \alpha^2 D_y^2)(\bar{f}\circ f - \kappa^2 \bar{g}\circ g) = 0. \tag{4.1b}$$

Let us consider the case $\alpha^2 = 1$, which corresponds to the metric tensor $g^{\mu\nu} = \mathrm{diag}(1, 1)$. Thus, by introducing the complex coordinates

$$\eta = x + \mathrm{i}y \qquad \bar{\eta} = x - \mathrm{i}y. \tag{4.2}$$

Equations (4.1) become

$$(\mathrm{i}D_t - 4D^2_{\eta\bar{\eta}})(\bar{f}\circ g) = 0 \tag{4.3a}$$

$$(\mathrm{i}D_t - 4D^2_{\eta\bar{\eta}})(\bar{f}\circ f - \kappa^2 \bar{g}\circ g) = 0. \tag{4.3b}$$

Confining ourselves to look for analytical solutions of equations (4.3), i.e. in the case in which the relations

$$\partial_{\bar{\eta}} g = 0 \qquad \partial_{\bar{\eta}} f = 0 \tag{4.4}$$

hold, we can find 'ghost solutions'. For instance, in the $su(2)$ case, we can choose

$$g = \mathrm{e}^{k\eta - \omega t + \delta} \qquad f = g\,\mathrm{e}^{-\mathrm{i}\gamma} \tag{4.5}$$

where γ, δ, ω and k are complex constants, with $\omega = 2|k|^2$. Then we get

$$\zeta = \mathrm{e}^{\mathrm{i}\gamma} \qquad S_3 = 0 \qquad S_+ = \frac{2e^{\mathrm{i}\gamma}}{1 + \kappa^2 \mathrm{e}^{-2\,\mathrm{Im}\,\gamma}} \qquad (v_1, v_2) = 4(\mathrm{Im}\,k, \mathrm{Re}\,k). \tag{4.6}$$

Another example is given by the choice

$$g = 0 \qquad f = \mathrm{e}^{k\eta - \omega\tau + \mathrm{i}\delta} \tag{4.7}$$

which leads to

$$\zeta = 0 \qquad S_3 = 1 \qquad S_+ = 0 \qquad (v_1, v_2) = 4(\mathrm{Im}\,k, \mathrm{Re}\,k). \tag{4.8}$$

Solutions of the form (4.6) and (4.8) can be used as asymptotic behaviour at infinity of solutions belonging to the topological sector.

In order to obtain vortex solutions, we make the choice

$$f = 1 \qquad \zeta = g. \tag{4.9}$$

Then, equations (4.1) reduce to

$$(\mathrm{i}\partial_t + \partial_x^2 + \alpha^2 \partial_y^2) g = 0 \tag{4.10a}$$

$$\partial_x \bar{g}\partial_x g + \alpha^2 \partial_y \bar{g}\partial_y g = 0 \tag{4.10b}$$

while the velocity field is given by

$$v_\mu = (2\mathrm{i}\kappa^2/(1 + \kappa^2|g|^2))(g\partial_\mu \bar{g} - \bar{g}\partial_\mu g). \tag{4.11}$$

Under the assumption (4.9), from (4.10*b*) it follows that non-trivial solutions exist only for pseudo-euclidean metric tensor, that is for $\alpha^2 = -1$. In this case and using the variables (4.2), the bilinear equations (4.1) read

$$(\mathrm{i}\partial_t + 2\partial_\eta^2 + 2\partial_{\bar\eta}^2)g = 0 \tag{4.12a}$$

$$\partial_\eta \bar g \partial_\eta g + \partial_{\bar\eta} \bar g \partial_{\bar\eta} g = 0. \tag{4.12b}$$

Equation (4.12*b*) is identically satisfied by any analytical function $g = g(\eta, t)$. Then, equation (4.12*a*) reduces to the one-dimensional time-dependent Schrödinger equation

$$(\mathrm{i}\partial_t + 2\partial_\eta^2)g = 0. \tag{4.13}$$

The simplest non-trivial solution of this equation is

$$g = \mathrm{e}^{k\eta - \omega t + \delta}, \tag{4.14}$$

where $k = k_1 + \mathrm{i}k_2$, $\omega = -2\mathrm{i}k^2$ and δ is a constant. The third spin component and the velocity field are given by

$$S_3 = 1 - \frac{2\kappa^2\rho_0^2}{\mathrm{e}^{-2\xi} + \kappa^2\rho_0^2} \qquad (v_1, v_2) = \frac{4\kappa^2\rho_0^2}{\mathrm{e}^{-2\xi} + \kappa^2\rho_0^2}(k_2, k_1) \tag{4.15}$$

where $\xi = k_1 x - k_2 y - 4k_1 k_2 t$ and $\rho_0 = \exp(\mathrm{Re}\,\delta)$. Concerning the asymptotic behaviour of the solution (4.15), we find that

$$\begin{aligned} &\text{for} \quad \xi \to -\infty \qquad S_3 \to 1 \qquad \text{and} \qquad (v_1, v_2) = (0, 0) \\ &\text{for} \quad \xi \to +\infty \qquad S_3 \to -1 \qquad \text{and} \qquad (v_1, v_2) = 4(k_2, k_1). \end{aligned} \tag{4.16}$$

This is a domain wall solution, located on the line $\xi = -\ln \rho_0$, where $S_3 = 0$ (in the compact case) or $S_3 \to \pm\infty$ (in the non-compact case). It propagates in the plane with velocity $\mathbf{v} = (4k_2, -4k_1)$. The vorticity for this solution is

$$\partial_x v_y - \partial_y v_x = 2\kappa^2(k_1^2 + k_2^2)\,\mathrm{sech}^2\,\xi \tag{4.17}$$

and the corresponding topological charge is divergent. A reason for this is that the asymptotic behaviour of the domain wall does not satisfy the compactification condition $S_3 \to 1$ for $x^2 + y^2 \to \infty$.

Following a well known procedure [27], we can expand the solution (4.15) in power series for $k_1 \ll 1$ and $R = k_2/k_1 = O(1)$. We have

$$g = g_0 \sum_{n=0}^{\infty} \frac{\varepsilon^n \eta^n}{n!} \sum_{m=0}^{\infty} \frac{\varepsilon^{2m}(2\mathrm{i}t)^m}{m!} = g_0 \sum_{N=0}^{\infty} \varepsilon^N \sum_{n+2m=N} (n!m!)^{-1} \eta^n (2\mathrm{i}t)^m \tag{4.18}$$

where $\eta = x + \mathrm{i}y$ and $\varepsilon = k_1(1 + \mathrm{i}R)$.

Truncating the series (4.18) to a given N, we obtain an exact time-dependent N-vortex solution. We can easily see that the corresponding topological charge is just N. In this sense a domain wall solution can be considered as a superposition of infinite number of vortices and, consequently, it possesses an infinite topological charge.

Because of the linearity of (4.13), the superposition of M domain walls

$$g = \sum_{j=1}^{M} \exp\{k_j \eta - \omega_j t + \delta_j\} \qquad \omega_{(j)} = -2\mathrm{i}k_{(j)}^2 \tag{4.19}$$

is also a solution of the same equation. Expanding, as we have done above, in power series one of the exponentials appearing in (4.19), we obtain interactions involving vortices and domain wall solutions.

Using the relations (4.9) (or more generally $\bar{f}=f$), from (3.8) the velocity field reduces to the form

$$v_\mu = -2i\kappa^2 \frac{\bar{\zeta}\partial_\mu \zeta - \zeta \partial_\mu \bar{\zeta}}{1+\kappa^2|\zeta|^2} \tag{4.20}$$

while (3.2a) can be written in terms of ζ only, namely

$$i\partial_t \zeta + \partial^\mu \partial_\mu \zeta - 2\kappa^2 \frac{\partial^\mu \bar{\zeta} \partial_\mu \zeta}{1+\kappa^2|\zeta|^2} \zeta = 0. \tag{4.21}$$

We notice that this result holds for any dimension D.

Finally, we recall that system (4.1) may provide solutions for the Ishimori model (see section 2). With this in mind, we need to know the scalar field ϕ in terms of the variables f and g. This can be done with the help of (3.8) and the expressions (2.2). So the auxiliary field ϕ is such that

$$\phi_y = 2i \frac{D_x(\bar{f}\circ f + \kappa^2 \bar{g}\circ g)}{|f|^2+\kappa^2|g|^2} \qquad \phi_x = 2i\alpha^2 \frac{D_y(\bar{f}\circ f + \kappa^2 \bar{g}\circ g)}{|f|^2+\kappa^2|g|^2}. \tag{4.22}$$

Furthermore, the integrability condition $\partial_x \phi_y = \partial_y \phi_x$ holds for $\alpha^2 = -1$. Exploiting the gauge equivalence technique, we can also find solutions of the Davey–Stewartson equation. For more details the reader is addressed to [9] and [19].8))

4.2. *Special solutions in $D=3$ dimensions*

In this subsection we show how to construct exact solutions with non-trivial vorticity tensor in three dimensions.

In doing so, we start again from (4.9). Adopting the metrics defined by $g^{\mu\nu} = \mathrm{diag}(1, \alpha^2, \beta^2)$, the bilinear equations (3.7) are reduced to the following system:

$$(i\partial_t + \partial_x^2 + \alpha^2 \partial_y^2 + \beta^2 \partial_z^2) g = 0 \tag{4.23a}$$

$$\partial_x \bar{g} \partial_x g + \alpha^2 \partial_y \bar{g} \partial_y g + \beta^2 \partial_z \bar{g} \partial_z g = 0. \tag{4.23b}$$

Interesting solutions to this system exist only in the case of pseudo-Euclidean metrics. For definiteness we choose $\alpha^2 = 1$, $\beta^2 = -1$.

Let us assume $g = \exp\{\mathbf{k}\cdot\mathbf{x} - \omega t + \delta\}$, where $\mathbf{k} = (k_1, k_2, k_3)$ is a complex three-dimensional wave vector, parametrized in the form

$$k_\mu = |k_\mu| \exp i\phi_\mu \qquad (\mu = 1, 2, 3), \tag{4.24a}$$

$$|k_1| = |k| \cos\lambda \qquad |k_2| = |k| \sin\lambda \qquad |k_3| = |k| \tag{4.24b}$$

with arbitrary real parameters λ and ϕ_μ. Then system (4.23) implies the dispersion relation

$$i\omega = |k|^2 \{\cos^2\lambda \exp 2i\phi_1 + \sin^2\lambda \exp 2i\phi_2 - \exp 2i\phi_3\}. \tag{4.25}$$

In analogy with the case $D=2$, we find the domain wall solutions moving in three dimensions, namely

$$S_+ = \frac{2g_0 \,\mathrm{e}^{-\xi+i\psi}}{\mathrm{e}^{-2\xi}+\kappa^2 g_0^2} \qquad S_3 = 1 - \frac{2\kappa^2 g_0^2}{\mathrm{e}^{-2\xi}+\kappa^2 g_0^2} \qquad v_\mu = \frac{4\kappa^2 g_0^2}{\mathrm{e}^{-2\xi}+\kappa^2 g_0^2} \,\mathrm{Im}\, k_\mu \tag{4.26}$$

where ξ and ψ are linear combinations of (x, y, z, t) given in terms of the λ and ϕ_μ's.

The asympotic behaviour of the component S_3 is $S_3 \to \pm 1$ for $\xi \to \mp\infty$.

In order to obtain rational solutions, we adopt the same procedure used above, expanding g in terms of $|k| \ll 1$:

$$g = g_0 \exp\{|k|a\} \exp\{|k|^2 ibt\} = g_0 \sum_{N=0}^{\infty} |k|^N \sum_{n+2m=N}^{N} (n!m!)^{-1} a^n (ibt)^m \quad (4.27)$$

where

$$\begin{cases} a = x\cos\lambda \exp i\phi_1 + y \sin\lambda \exp i\phi_2 + z \exp i\phi_3 \\ b = \cos^2\lambda \exp 2i\phi_1 + \sin^2\lambda \exp 2i\phi_2 - \exp 2i\phi_3. \end{cases} \quad (4.28)$$

For $N=1$ we get static solutions. In this case the velocity field components are

$$\begin{cases} v_1 = (4v_0/\Delta)\{y \sin\lambda\cos\lambda \sin(\phi_1-\phi_2) + z\cos\lambda \sin(\phi_1-\phi_3)\} \\ v_2 = (4v_0/\Delta)\{-x\sin\lambda\cos\lambda\sin(\phi_1-\phi_2) + z\sin\lambda\sin(\phi_2-\phi_3)\} \\ v_3 = (4v_0/\Delta)\{-x\cos\lambda\sin(\phi_1-\phi_3) - y\sin\lambda\sin(\phi_2-\phi_3)\} \end{cases} \quad (4.29)$$

where

$$\begin{cases} \Delta = 1 + v_0\{x^2\cos^2\lambda + y^2\sin^2\lambda + z^2 + 2xy\sin\lambda\cos\lambda\cos(\phi_1-\phi_2) \\ \qquad + 2xz\cos\lambda\cos(\phi_1-\phi_3) + 2yz\sin\lambda\cos(\phi_2-\phi_3)\} \\ v_0 = \kappa^2 |g_0|^2 |k|^2. \end{cases} \quad (4.30)$$

The spin field $\mathbf{S}$ can be expressed by (3.4) with the help of (4.27) for $N=1$. By direct calculation we find that the vorticity tensor has non-vanishing components in any direction. In general, for an arbitrary N, we obtain time-dependent rational solutions of the vortex type in three dimensions.

Now we observe that in our context it is possible to obtain a relativistic vortex dynamics in $2+1$ dimensions. To this aim we consider static solution of the $3+1$-dimensional version of (2.1) and regard the third coordinate z as a new time variable τ. The equations of motion read

$$\begin{cases} 2i(v_a \partial_a S - v_0 \partial_\tau S) = [S, \Delta S] - [S, \partial_\tau^2 S] \qquad (a=1,2) \\ \partial_\mu v_\nu - \partial_\nu v_\mu = -iS[\partial_\mu S, \partial_\nu S] \qquad (\mu,\nu = 0,1,2) \end{cases} \quad (4.31)$$

which describe a relativistic nonlinear σ-model for the spin matrix

$$S = \begin{pmatrix} S_3 & \kappa\bar{S}_+ \\ \kappa S_+ & -S_3 \end{pmatrix}. \quad (4.32)$$

Using the relations (4.9) and the complex variable η (see (4.2)), the bilinear equations corresponding to system (4.31) are reduced to the form

$$\begin{cases} \partial_\tau^2 g - 4\partial_{\eta\bar\eta}^2 g = 0 \\ 2(\partial_\eta \bar g \partial_{\bar\eta} g + \partial_{\bar\eta}\bar g \partial_\eta g) = \partial_\tau \bar g \partial_\tau g. \end{cases} \quad (4.33)$$

Choosing the particular solution to this system of the form

$$g = a\eta + \sqrt{2}a\, e^{i\alpha}\tau \quad (4.34)$$

we are led to a moving vortex solution with topological charge $N=1$.

Another interesting example of exact solutions comes from the choice

$$g = g_0 \exp[\chi(e^{i\alpha}\eta + e^{i\beta}\bar{\eta} - 2e^{i(\alpha+\beta)/2}\tau)] \tag{4.35}$$

where $\chi > 0$, α and β are real constants. Resorting to an expansion in χ analogous to (4.18), we arrive at time-dependent vortex–antivortex configurations. The simplest examples of such solutions correspond to the following expressions for g:

$$g = g_0\chi\{e^{i\alpha}\eta + e^{i\beta}\bar{\eta} - 2e^{i(\alpha+\beta)/2}\tau\} \qquad \text{for } N=1 \tag{4.36}$$

and

$$g = g_0 2^{-1}\chi^2\{e^{i\alpha}\eta + e^{i\beta}\bar{\eta} - 2e^{i(\alpha+\beta)/2}\tau\}^2 \qquad \text{for } N=2. \tag{4.37}$$

Inserting these quantities into (3.1) and (3.8), we can build up the explicit expressions for the spin field **S** and the velocity field v_μ.

4.3. *Special solutions in higher dimensions*

Here we discuss some simple solutions in an arbitrary number of space dimensions. First, let us consider the case in which the space has an even number of dimensions $D=2n$ and is endowed with the metrics $g^{\mu\nu} = \mathrm{diag}(I_n, -I_n)$. Then we can exploit the bilinear equations (3.7), by introducing n complex coordinates $\eta_l = x_l + ix_{l+1}$ $(l=1,\ldots,n)$, and the functions f and g as in (4.9). Classes of particular solutions can be found by limiting ourselves to the case in which g is an analytical function (i.e. $\partial_{\bar{\eta}_l} g = 0$, $(l=1,\ldots,n)$). In this way we are led to the n-dimensional linear Schrödinger equation for a free particle, which is a generalization of (4.13). Therefore we can extend the results achieved above, obtaining multidimensional domain wall solutions and superposition of domain walls and rational solutions with non-trivial vorticity.

Second, in the case $D=2n+1$ with $g^{\mu\nu} = \mathrm{diag}(+I_{2n}, -1)$ we consider the static solutions of (2.1). Regarding the $(2n+1)$th coordinate as a new time variable τ, we get an extension of system (4.31). Consequently, using again (4.9), the relations corresponding to (4.32) read

$$\begin{gathered} \partial_\tau^2 g - 4\sum_{l=1}^{n} \partial^2_{\eta_l\bar{\eta}_l} g = 0 \\ 2\sum_{l=1}^{n} \partial_{\eta_l}\bar{g}\partial_{\bar{\eta}_l} g + \partial_{\bar{\eta}_l}\bar{g}\partial_{\eta_l} g) = \partial_\tau\bar{g}\partial_\tau g. \end{gathered} \tag{4.38}$$

These equations admit classes of solutions which suitably generalize the expressions (4.34) and (4.35). So we can provide topological solutions of vortex and vortex–antivortex type in $2n$ space dimensions.

Finally, it is a remarkable fact that we can find spin-wave solutions for system (2.1) in any space dimension D. Indeed, the functions

$$g = \rho_0 \exp\left\{ i\left(\sum_{\mu=1}^{D} k_\mu x_\mu - \omega t + \omega_0\right)\right\} \qquad f = \exp\left\{ i\left(\sum_{\mu=1}^{D} p_\mu x_\mu - \varphi t\right)\right\} \tag{4.39}$$

where

$$\begin{cases} k_\mu = (1+\kappa^2\rho_0^2)^{-1}P_{0\mu} & p_\mu = -\kappa^2\rho_0^2(1+\kappa^2\rho_0^2)^{-1}P_{0\mu} \\ \omega = (1-\kappa^2\rho_0^2)(1+\kappa^2\rho_0^2)^{-2}P_0^2 & \varphi = -\kappa^2\rho_0^2(1-\kappa^2\rho_0^2)(1+\kappa^2\rho_0^2)^{-2}P_0^2 \end{cases} \tag{4.40}$$

are solutions of the bilinear equations (3.7). The corresponding stereographic projection ζ is given by

$$\zeta=\rho_0\exp\left\{\mathrm{i}\left(\sum_{\mu=1}^{D}P_{0\mu}x_\mu-\Omega_0 t+\omega_0\right)\right\}\tag{4.41}$$

where $\mathbf{P}_0=\mathbf{k}-\mathbf{p}$, $\Omega_0=\omega-\varphi$ and the dispersion relation

$$\Omega_0=\frac{1-\kappa^2\rho_0^2}{1+\kappa^2\rho_0^2}P_0^2+P_0^\mu v_{0\mu}=S_3P_0^2+P_0^\mu v_{0\mu}\tag{4.42}$$

holds. Now one can show that for the $SU(1,1)$ model in $D=1$ and for the corresponding Ishimori-II version ($D=2$ and $g^{\mu\nu}=\mathrm{diag}(1,-1)$), the dispersion relation (4.42) can be suitably modified, assuming that the third component S_3 of the spin field depends on the wave number $\mathbf{P}_0$ in a physically meaningful way. In order to be brief we shall skip all details (which will be presented in [19]), here we claim that this possibility is assured by the gauge equivalence between the above-mentioned systems and the nonlinear Schrödinger equation of the repulsive type (NLS_-) for $D=1$, or the Davey–Stewartson equation ($D=2$), which provides, under proper boundary conditions, a description of a repulsive Bose gas (see the review article about spin models in [1]). For instance, a one-dimensional Bose gas, whose particle density at infinity is $E_0=\rho$, can be described by solutions of NLS_- which asymptotically tend to $\sqrt{\rho}$. The gauge equivalence theory associates this type of solutions with spin-wave solutions of the Landau–Lifshitz equation (the one-dimensional reduction of system (2.1)), in such a way that the corresponding density of energy

$$E_0=\frac{\bar{\zeta}_x\zeta_x}{(1-|\zeta|^2)^2}\tag{4.43}$$

is equal to the density of particles in the former case. From (4.41), the above-mentioned relatonship between densities can be satisfied only if the amplitude ρ_0 will depend on the wavenumber P_0, precisely

$$\rho_0^2=1-\frac{P_0^2}{2\rho}((1+4\rho/P_0^2)^{1/2}-1).\tag{4.44}$$

In the upper half-plane of the pseudosphere $S^{1,1}$, the third component of the spin field is

$$S_3=\frac{1+\rho_0^2}{1-\rho_0^2}=\frac{(P_0^2+4\rho)^{1/2}}{P_0}\tag{4.45}$$

and the frequency Ω_0 has the Bogolyubov form [1]

$$\Omega_0=P_0(P_0^2+4\rho)^{1/2}.\tag{4.46}$$

These solutions are known as 'hole-like' spin-wave [1].

For the Ishimori-II model, the situation is much more intricate. However, following the same route, we are led to the dispersion relation

$$\Omega_0=\pm(P_{01}^2-P_{02}^2)\frac{(P_{01}^2+P_{02}^2-4\rho)^{1/2}}{(P_{01}^2+P_{02}^2)^{1/2}}\tag{4.47}$$

which is different from the two-dimensional Bogolyubov dispersion relation.

Nevertheless, (4.47) reduces to (4.46) when one of the wavenumber components vanishes. Finally, for $|P_{01}|=|P_{02}|$, equation (4.47) provides a zero-mode (or 'Goldstone mode') solution.

5. Conclusions

We have proposed and analysed a multidimensional spin field model endowed with both a compact and a non-compact symmetry. The main feature possessed by the equations of motion is the coupling between the spin field and the velocity field, whose vorticity is connected with the topological current density.

The model introduced particularly modifies some well known topological nonlinear field systems; a few of them turn out to be linearizable by a Laxa formulation and admit exact solutions mimicking a particle-like behaviour. Although most of these solvable models arise from mathematical speculations, they might be a useful guide to build up more realistic field theories. This task could be made easier by a more complete study of (2.1), whose unifying character allows us to tackle globally many problems inherent to a whole class of topological nonlinear field models [19].

It should be also noted that (2.1) offer the possibility of clarifying the role of the symmetry structure of the spin phase space. In fact, in the cases where the spin field variables range over a compact or a noncompact manifold, equations (2.1) may lead to solutions with different properties.

To conclude, we observe that the approach based on the bilinearization technique reveals a powerful tool for handling nonlinear field equations such as (2.1). The results obtained produce new classes of exact solutions. Here we were concerned mainly with the topological non-trivial solutions. Nevertheless, by the same techniques one could construct exponentially localized soliton solutions, which will be considered in a future paper.

Finally, an important aspect regarding the model (2.1) is that it can cover a broad range of topics, from the propagation of domain walls, spin waves and magnetic vortices in condensed matter, to nonlinear field theories in high energy physics.

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